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INTEGRAL TRANSFORM OF CERTAIN SUBCLASSES OF UNIVALENT FUNCTIONS

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ABSTRACT: *In this paper, we study a new class of functions defined by Wright generalized hypergeometric function. Characterization property, the result on modified Hadamard product and integral transform are obtained. Distortion theorem and radii of starlikeness and convexity are also obtained.*

KEY WORDS: *Analytic function, Uniformly convex, Wright generalized hypergeometric function, Linear operator , Hadamard product.*

I. INTRODUCTION

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (a_n \geq 0) \tag{1.1}$$

which are analytic and univalent in the unit disk $U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$.
A function $f \in A$ is said to be in the class of uniformly convex functions of order γ, denoted by $UCV(\gamma)$(cf. [5]) if

$$\text{Re}\left\{\frac{zf''(z)}{f'(z)} + 1 - \gamma\right\} \geq \eta\left|\frac{zf''(z)}{f'(z)} - 1\right|, \tag{1.2}$$

and is said to be in a corresponding subclass of $UCV(\gamma)$ denoted by $S_p(\gamma)$ if

$$\text{Re}\left\{\frac{zf'(z)}{f(z)} - \gamma\right\} \geq \eta\left|\frac{zf'(z)}{f(z)} - 1\right|, \tag{1.3}$$

where $-1 \leq \gamma \leq 1$ and $z \in U$.
The class of uniformly convex and uniformly starlike function has been studied by Goodman ([1], [2]) and Ma and Minda [11].
If $f(z)$ of the form (1.1) in class A and function $g(z) \in A$ defined as

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n, \tag{1.4}$$

then the Hadamard product of $f(z)$ and $g(z)$ is given by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n . \tag{1.5}$$

Let T denote the subclass of A consisting of functions of the form (cf. [7])

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \qquad (a_n \geq 0). \tag{1.6}$$

A function $\psi_p\left[\left(\alpha_j A_j\right)_{1,q}; \left(\beta_j B_j\right)_{1,s}; z\right]$ is studied by R.K. Raina [9] as

$$\psi_p\left[\left(\alpha_j A_j\right)_{1,q}; \left(\beta_j B_j\right)_{1,s}; z\right] = \bar{\omega} z\, {}_q\psi_s\left[\left(\alpha_j A_j\right)_{1,q}; \left(\beta_j B_j\right)\right], \tag{1.7}$$

where

$$\bar{\omega} = \frac{\prod_{j=1}^{s} \Gamma(\beta_j)}{\prod_{j=1}^{q} \Gamma(\alpha_j)}, (q, s \in N_0) \tag{1.8}$$

and ${}_q\psi_s\left[\left(\alpha_j A_j\right)_{1,q}; \left(\beta_j B_j\right)_{1,s}; z\right]$ is the Wright generalized hypergeometric function introduced by Wright [3] as

$$ {}_q\psi_s\left[\left(\alpha_j A_j\right)_{1,q};\left(\beta_j B_j\right)_{1,s};z\right]=\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{q}\Gamma\left(\alpha_j+A_j n\right)z^n}{\prod_{j=1}^{s}\Gamma\left(\beta_j+B_j n\right)(n)!}\quad,\qquad (z\in U)\qquad (1.9)$$

by making use of the Hadamard product, Raina [9] defined a linear operator $\Xi_p\left[\left(\alpha_j A_j\right)_{1,q};\left(\beta_j B_j\right)_{1,s}\right]f(z):T\to T$ as

$$\Xi_p\left[\left(\alpha_j A_j\right)_{1,q};\left(\beta_j B_j\right)_{1,s}\right]f(z)=\psi_p\left[\left(\alpha_j A_j\right)_{1,q};\left(\beta_j B_j\right)_{1,s};z\right]*f(z)$$
$$=z-\sum_{n=2}^{\infty}\sigma_n\,a_n z^n\,,\qquad (1.10)$$

where

$$\sigma_n=\frac{\bar{\omega}\prod_{j=1}^{q}\Gamma(\alpha_j+A_j(n-1))}{\prod_{j=1}^{s}\Gamma\left(\beta_j+B_j(n-1)\right)\;(n-1)!}\qquad (1.11)$$

and $\bar{\omega}$ is defined by equation (1.8).

Definition 1. A function $f\in A$ for $-1\le\gamma<1$ is said to be in the class $T^*(\gamma)$ if and only if

$$\mathrm{Re}\left[\frac{z\left(\Xi_p\left(\left(\alpha_j A_j\right)_{1,q};\left(\beta_j B_j\right)_{1,s}\right)f(z)\right)'}{\left(\Xi_p\left(\left(\alpha_j A_j\right)_{1,q};\left(\beta_j B_j\right)_{1,s}\right)f(z)\right)}-\gamma\right]\ge\left|\frac{z\left(\Xi_p\left(\left(\alpha_j A_j\right)_{1,q};\left(\beta_j B_j\right)_{1,s}\right)f(z)\right)'}{\left(\Xi_p\left(\left(\alpha_j A_j\right)_{1,q};\left(\beta_j B_j\right)_{1,s}\right)f(z)\right)}-1\right|,\ z\in U.\quad (1.12)$$

Definition 2. A function $f\in A$ for $-1\le\gamma<1$ is said to be in the class $UCVT^*(\gamma)$ if and only if

$$\mathrm{Re}\left[1+\frac{z\left(\Xi_p\left(\left(\alpha_j A_j\right)_{1,q};\left(\beta_j B_j\right)_{1,s}\right)f(z)\right)''}{\left(\Xi_p\left(\left(\alpha_j A_j\right)_{1,q};\left(\beta_j B_j\right)_{1,s}\right)f(z)\right)'}-\gamma\right]\ge\left|\frac{z\left(\Xi_p\left(\left(\alpha_j A_j\right)_{1,q};\left(\beta_j B_j\right)_{1,s}\right)f(z)\right)''}{\left(\Xi_p\left(\left(\alpha_j A_j\right)_{1,q};\left(\beta_j B_j\right)_{1,s}\right)f(z)\right)'}-1\right|,\ z\in U.\quad (1.13)$$

II. CHARACTERIZATION PROPERTY

Theorem 2.1. A function $f(z)$ defined by (1.6) is in the class $T^*(\gamma)$ if and only if

$$\sum_{n=2}^{\infty}(2n-1-\gamma)\sigma_n a_n\le 1-\gamma,(-1\le\gamma<1).\qquad (2.1)$$

The result is sharp for the function

$$f(z)=z-\frac{(1-\gamma)}{(2n-1-\gamma)\sigma_n}z^n,$$

where $\sigma_n=\dfrac{\prod_{j=1}^{s}\Gamma\left(\beta_j\right)\prod_{j=1}^{q}\Gamma(\alpha_j+A_j(n-1))}{\prod_{j=1}^{q}\Gamma\left(\alpha_j\right)\prod_{j=1}^{s}\Gamma\left(\beta_j+B_j(n-1)\right)(n-1)!}.$

Proof. It is sufficient to show that

$$\left|\frac{z\left(\Xi_p\left(\left(\alpha_j A_j\right)_{1,q};\left(\beta_j B_j\right)_{1,s}\right)f(z)\right)'}{\left(\Xi_p\left(\left(\alpha_j A_j\right)_{1,q};\left(\beta_j B_j\right)_{1,s}\right)f(z)\right)}-1\right| \leq \operatorname{Re}\left\{\frac{z\left(\Xi_p\left(\left(\alpha_j A_j\right)_{1,q};\left(\beta_j B_j\right)_{1,s}\right)f(z)\right)'}{\left(\Xi_p\left(\left(\alpha_j A_j\right)_{1,q};\left(\beta_j B_j\right)_{1,s}\right)f(z)\right)}-\gamma\right\},$$

$$\left|\frac{z\left(\Xi_p\left(\left(\alpha_j A_j\right)_{1,q};\left(\beta_j B_j\right)_{1,s}\right)f(z)\right)'}{\left(\Xi_p\left(\left(\alpha_j A_j\right)_{1,q};\left(\beta_j B_j\right)_{1,s}\right)f(z)\right)}-1\right| - \operatorname{Re}\left\{\frac{z\left(\Xi_p\left(\left(\alpha_j A_j\right)_{1,q};\left(\beta_j B_j\right)_{1,s}\right)f(z)\right)'}{\left(\Xi_p\left(\left(\alpha_j A_j\right)_{1,q};\left(\beta_j B_j\right)_{1,s}\right)f(z)\right)}-1\right\} \leq 1-\gamma$$

we have

$$\left|\frac{z\left(\Xi_p\left(\left(\alpha_j A_j\right)_{1,q};\left(\beta_j B_j\right)_{1,s}\right)f(z)\right)'}{\left(\Xi_p\left(\left(\alpha_j A_j\right)_{1,q};\left(\beta_j B_j\right)_{1,s}\right)f(z)\right)}-1\right| - \operatorname{Re}\left\{\frac{z\left(\Xi_p\left(\left(\alpha_j A_j\right)_{1,q};\left(\beta_j B_j\right)_{1,s}\right)f(z)\right)'}{\left(\Xi_p\left(\left(\alpha_j A_j\right)_{1,q};\left(\beta_j B_j\right)_{1,s}\right)f(z)\right)}-1\right\}$$

$$\leq 2\left|\frac{z\left(\Xi_p\left(\left(\alpha_j A_j\right)_{1,q};\left(\beta_j B_j\right)_{1,s}\right)f(z)\right)'}{\left(\Xi_p\left(\left(\alpha_j A_j\right)_{1,q};\left(\beta_j B_j\right)_{1,s}\right)f(z)\right)}-1\right|$$

$$\leq 2\left|\frac{1-\sum_{n=2}^{\infty} n\sigma_n a_n z^{n-1}}{1-\sum_{n=2}^{\infty} \sigma_n a_n z^{n-1}}-1\right|$$

or,

$$\leq 2\left|\frac{\sum_{n=2}^{\infty} (n-1)\sigma_n a_n z^{n-1}}{1-\sum_{n=2}^{\infty} \sigma_n a_n z^{n-1}}\right| \leq 1-\gamma.$$

The above inequality must hold for all z in U. Letting $z \to 1^-$, we have

$$\leq \frac{2\sum_{n=2}^{\infty} (n-1)\sigma_n a_n}{1-\sum_{n=2}^{\infty} \sigma_n a_n} \leq (1-\gamma).$$

Therefore,

$$\sum_{n=2}^{\infty}(2n-1-\gamma)\sigma_n a_n \leq (1-\gamma).$$

Conversely, if $f \in T^*(\gamma)$ and z is real then (2.1) gives

$$\frac{1-\sum_{n=2}^{\infty} n\sigma_n a_n z^{n-1}}{1-\sum_{n=2}^{\infty} \sigma_n a_n z^{n-1}} - \gamma \geq \frac{\sum_{n=2}^{\infty} (n-1)\sigma_n a_n z^{n-1}}{1-\sum_{n=2}^{\infty} \sigma_n a_n z^{n-1}},$$

by letting $z \to 1^-$ along the real axis, we get

$$\frac{1-\sum_{n=2}^{\infty} n\sigma_n a_n}{1-\sum_{n=2}^{\infty} \sigma_n a_n} - \frac{\sum_{n=2}^{\infty} (n-1)\sigma_n a_n}{1-\sum_{n=2}^{\infty} \sigma_n a_n} \geq \gamma,$$

$$1-\sum_{n=2}^{\infty}(2n-1)\,\sigma_n a_n \geq (1-\gamma)\left(1-\sum_{n=2}^{\infty}\sigma_n a_n\right),$$

which yields the required result, where σ_n is given by (1.11).

Corollary 2.1. Let the function $f(z)$ defined by (1.6) be in the class $T^*(\gamma)$, then

$$a_n \leq \frac{(1-\gamma)}{(2n-1-\gamma)\sigma_n}, n \geq 2,$$

where

$$\sigma_n = \frac{\prod_{j=1}^{s} \Gamma(\beta_j) \prod_{j=1}^{q} \Gamma(\alpha_j + A_j(n-1))}{\prod_{j=1}^{q} \Gamma(\alpha_j) \prod_{j=1}^{s} \Gamma(\beta_j + B_j(n-1))\,(n-1)!}, \qquad n \geq 2.$$

III. Growth and Distortion theorems

Theorem II.1. Let the function $f(z)$ defined by (1.6) be in the class $T^*(\gamma)$, then

$$|z| - \left(\tfrac{1-\gamma}{3-\gamma}\right)|z|^2 \leq \left|\left(\Xi_p\left[(\alpha_j A_j)_{1,q};(\beta_j B_j)_{1,s}\right]f(z)\right)\right| \leq |z| + \left(\tfrac{1-\gamma}{3-\gamma}\right)|z|^2 \qquad (3.1)$$

and

$$1 - 2\left(\tfrac{1-\gamma}{3-\gamma}\right)|z| \leq \left|\left(\Xi_p\left[(\alpha_j A_j)_{1,q};(\beta_j B_j)_{1,s}\right]f(z)\right)'\right| \leq 1 + 2\left(\tfrac{1-\gamma}{3-\gamma}\right)|z|. \qquad (3.2)$$

Equality holds for the function

$$f(z) = z - \left(\frac{1-\gamma}{3-\gamma}\right)\frac{z^2}{\sigma_2},$$

where $\sigma_2 = \dfrac{\prod_{j=1}^{s} \Gamma(\beta_j) \prod_{j=1}^{q} \Gamma(\alpha_j + A_j)}{\prod_{j=1}^{q} \Gamma(\alpha_j) \prod_{j=1}^{s} \Gamma(\beta_j + B_j)}, \; n \geq 2.$ (3.3)

Proof. Let $f(z) \in T^*(\gamma)$. By using(1.10), we get

$$\left|\Xi_p\left[(\alpha_j A_j)_{1,q};(\beta_j B_j)_{1,s}\right]f(z)\right| \le |z| + \sum_{n=2}^{\infty}\sigma_n\, a_n |z|^n,$$

by using (2.1) i.e.

$$\sum_{n=2}^{\infty}\sigma_n\, a_n \le \left(\frac{1-\gamma}{2n-1-\gamma}\right) = \left(\frac{1-\gamma}{3-\gamma}\right), n \ge 2,$$

then, we obtain

$$\left|\left(\Xi_p\left[(\alpha_j A_j)_{1,q};(\beta_j B_j)_{1,s}\right]f(z)\right)\right| \le |z| + |z|^2\sum_{n=2}^{\infty} a_n\sigma_n,$$

$$\left|\left(\Xi_p\left[(\alpha_j A_j)_{1,q};(\beta_j B_j)_{1,s}\right]f(z)\right)\right| \le |z|\left\{1+\left(\frac{1-\gamma}{3-\gamma}\right)|z|\right\}, \tag{3.4}$$

and $\left|\left(\Xi_p\left[(\alpha_j A_j)_{1,q};(\beta_j B_j)_{1,s}\right]f(z)\right)\right| \ge |z|\left\{1-\left(\frac{1-\gamma}{3-\gamma}\right)|z|\right\}.$ (3.5)

Similarly $\left|\left(\Xi_p\left[(\alpha_j A_j)_{1,q};(\beta_j B_j)_{1,s}\right]f(z)\right)'\right| \ge 1-2\left(\frac{1-\gamma}{3-\gamma}\right)|z|,$ (3.6)

And $\left|\left(\Xi_p\left[(\alpha_j A_j)_{1,q};(\beta_j B_j)_{1,s}\right]f(z)\right)'\right| \le 1+2\left(\frac{1-\gamma}{3-\gamma}\right)|z|.$ (3.7)

By using (3.4) and (3.5), we obtain

$$|z| - \left(\frac{1-\gamma}{3-\gamma}\right)|z|^2 \le \left|\left(\Xi_p\left[(\alpha_j A_j)_{1,q};(\beta_j B_j)_{1,s}\right]f(z)\right)\right| \le |z| + \left(\frac{1-\gamma}{3-\gamma}\right)|z|^2,$$

and by using (3.6) and (3.7), we get

$$1-2\left(\frac{1-\gamma}{3-\gamma}\right)|z| \le \left|\left(\Xi_p\left[(\alpha_j A_j)_{1,q};(\beta_j B_j)_{1,s};z\right]f(z)\right)'\right| \le 1+2\left(\frac{1-\gamma}{3-\gamma}\right)|z|.$$

The bounds (3.1) and (3.2) are attained for functions $f(z)$ given by

$$f(z) = z - \frac{(1-\gamma)}{(3-\gamma)\sigma_2}z^2,$$

where σ_2 is given by (3.3).

Theorem III.2 Let the function $f(z)$ defined by (1.6) and

$$g(z) = z - \sum_{n=2}^{\infty} b_n z^n \tag{3.8}$$

be in the class $T^*(\gamma)$. Then the function $h(z)$ defined by

$$h(z) = (1-\lambda)f(z) + \lambda g(z) = z - \sum_{n=2}^{\infty} q_n z^n, \tag{3.9}$$

where $q_n = (1-\lambda)a_n + \lambda b_n$, $(0 \le \lambda \le 1)$ is also in class $T^*(\gamma)$

Proof. By using (2.1) for $f(z)$ and $g(z)$, we have

$$\sum_{n=2}^{\infty}(2n-1-\gamma)\sigma_n a_n \le (1-\gamma) \tag{3.10}$$

And

$$\sum_{n=2}^{\infty}(2n-1-\gamma)\sigma_n b_n \le (1-\gamma). \tag{3.11}$$

On using (3.10) and (3.11) in (3.9), we get

$$h(z) = (1-\lambda)\left(z - \sum_{n=2}^{\infty} a_n z^n\right) + \lambda\left(z - \sum_{n=2}^{\infty} b_n z^n\right),$$

$$= z - \sum_{n=2}^{\infty}[(1-\lambda)\,a_n + \lambda b_n]z^n,$$

then

$$\sum_{n=2}^{\infty}(2n-1-\gamma)\sigma_n[(1-\lambda)a_n+\lambda b_n]=\sum_{n=2}^{\infty}(2n-1-\gamma)(1-\lambda)\sigma_n a_n+\lambda\sum_{n=2}^{\infty}(2n-1-\gamma)\sigma_n b_n,$$

$$\leq (1-\lambda)(1-\gamma)+\lambda(1-\gamma),$$

$$\leq (1-\gamma).$$

So $h(z)$ is also in the class $T^*(\gamma)$.

IV CLOSURE THEOREM

Theorem 4.1. Let the functions

$$f_j(z)=z-\sum_{n=2}^{\infty}a_{n,j}\,z^n\,,(j=1,2,\dots\dots\dots m) \tag{4.1}$$

be in the classes $T^*(\gamma_j)$ $(j=1,2,\dots\dots\dots m)$ respectively. Then the function $h(z)$ defined by

$$h(z)=z-\frac{1}{m}\sum_{n=2}^{\infty}\left(\sum_{j=1}^{m}a_{n,j}\right)z^n \tag{4.2}$$

is in the class $T^*(\gamma)$, where

$$\gamma=\min_{1\leq j\leq m}\{\gamma_j\},\quad \text{with } 0\leq\gamma_j<1. \tag{4.3}$$

Proof. Since $f_j\in T^*(\gamma_j)$, $(j=1,2,\dots\dots\dots m)$. By using (2.1) in (4.2), we get

$$\sum_{n=2}^{\infty}\sigma_n(2n-1-\gamma)\left(\frac{1}{m}\sum_{j=1}^{m}a_{n,j}\right)=\frac{1}{m}\sum_{j=1}^{m}\left(\sum_{n=2}^{\infty}(2n-1-\gamma)\sigma_n\,a_{n,j}\right)$$

$$\leq\frac{1}{m}\sum_{j=1}^{m}(1-\gamma_j)\leq 1-\gamma[\text{by using (4.3)}].$$

Hence $h\in T^*(\gamma)$ so the proof is completed.

V Results involving modified Hadamard product

Let $f(z)$ and $g(z)$ defined by (1.6) and (3.8) respectively. Then

$$(f*g)(z)=z-\sum_{n=2}^{\infty}a_n b_n z^n$$

be the modified Hadamard product of function $f(z)$ and $g(z)$.

Theorem 5.1. For functions

$$f_j(z)=z-\sum_{n=2}^{\infty}a_{n,j}\,z^n,\left(a_{n,j}\geq 0;z\in U;(j=1,2)\right),$$

let $f_1(z)\in T^*(\gamma)$ and $f_2(z)\in T^*(\eta)$. Then $(f_1*f_2)(z)\in T^*(\xi)$,

where
$$\xi=1-\frac{2(1-\gamma)(1-\beta)}{(3-\gamma)(3-\beta)\sigma_2-(1-\gamma)(1-\beta)} \tag{5.1}$$

and
$$\sigma_2=\frac{\prod_{j=1}^{s}\Gamma(\beta_j)\prod_{j=1}^{q}\Gamma(\alpha_j+A_j)}{\prod_{j=1}^{q}\Gamma(\alpha_j)\prod_{j=1}^{s}\Gamma(\beta_j+B_j)}. \tag{5.2}$$

The result is best possible for the functions

$$f_1=z-\frac{1-\gamma}{(3-\gamma)\sigma_2}z^2, \tag{5.3}$$

$$f_2 = z - \frac{1-\eta}{(3-\eta)\sigma_2} z^2. \tag{5.4}$$

Proof . By using (2.1), it is sufficient to prove that

$$\sum_{n=2}^{\infty} \frac{2n-1-\xi}{1-\xi} \sigma_n \left(a_{n,1}\, a_{n,2}\right) \leq 1, \tag{5.5}$$

where ξ is defined by (5.1) under the hypothesis, it follows from (2.1)

$$\sum_{n=2}^{\infty} \frac{2n-1-\gamma}{1-\gamma} \sigma_n a_{n,1} \leq 1 \tag{5.6}$$

and

$$\sum_{n=2}^{\infty} \frac{2n-1-\eta}{1-\eta} \sigma_n a_{n,2} \leq 1. \tag{5.7}$$

By using Cauchy-Schwarz inequality

$$\sum_{n=2}^{\infty} \frac{(2n-1-\gamma)^{1/2}(2n-1-\eta)^{1/2}}{\sqrt{(1-\gamma)(1-\eta)}} \sigma_n \sqrt{\left(a_{n,1} a_{n,2}\right)} \leq 1, \tag{5.8}$$

thus, to find largest ξ such that

$$\sum_{n=2}^{\infty} \frac{2n-1-\xi}{1-\xi} \sigma_n a_{n,1} a_{n,2} \leq \sum_{n=2}^{\infty} \frac{(2n-1-\gamma)^{1/2}(2n-1-\eta)^{1/2}}{\sqrt{(1-\gamma)(1-\eta)}} \sigma_n \sqrt{\left(a_{n,1} a_{n,2}\right)} \leq 1,$$

$$\sqrt{\left(a_{n,1} a_{n,2}\right)} \leq \frac{(2n-1-\gamma)^{1/2}\,(2n-1-\eta)^{1/2}\,(1-\xi)}{\sqrt{(1-\gamma)(1-\eta)}\,(2n-1-\xi)}, \text{ for } n \geq 2 \tag{5.9}$$

then (5.8) reduces to,

$$\sqrt{\left(a_{n,1} a_{n,2}\right)} \leq \frac{\sqrt{(1-\gamma)(1-\eta)}}{(2n-1-\gamma)^{1/2}\,(2n-1-\eta)^{1/2}\,\sigma_n}, \quad \text{for } n \geq 2 \tag{5.10}$$

it is sufficient to find the largest σ such that

$$\frac{\sqrt{(1-\gamma)(1-\eta)}}{(2n-1-\gamma)^{1/2}\,(2n-1-\eta)^{1/2}\,\sigma_n} \leq \frac{(2n-1-\gamma)^{1/2}\,(2n-1-\eta)^{1/2}\,(1-\xi)}{\sqrt{(1-\gamma)(1-\eta)}\,(2n-1-\xi)}, \text{for } n \geq 2$$

$$\frac{2n-1-\xi}{1-\xi} \leq \frac{(2n-1-\gamma)(2n-1-\eta)\sigma_n}{(1-\gamma)(1-\eta)}, \tag{5.11}$$

$$\xi[(2n-1-\gamma)(2n-1-\eta)\sigma_n - (1-\gamma)(1-\eta)]$$

$$\leq (2n-1-\gamma)(2n-1-\eta)\sigma_n - (2n-2+1)(1-\gamma)(1-\eta),$$

$$\leq [(2n-1-\gamma)(2n-1-\eta)\sigma_n - (1-\gamma)(1-\eta)] - 2(n-1)(1-\gamma)(1-\eta),$$

$$\xi \leq 1 - \frac{2(n-1)(1-\gamma)(1-\eta)}{(2n-1-\gamma)(2n-1-\eta)\sigma_n - (1-\gamma)(1-\eta)},$$

where

$$\sigma_n = \frac{\prod_{j=1}^{s} \Gamma(\beta_j) \prod_{j=1}^{q} \Gamma(\alpha_j + A_j(n-1))}{\prod_{j=1}^{q} \Gamma(\alpha_j) \prod_{j=1}^{s} \Gamma(\beta_j + B_j(n-1))\,(n-1)!} \text{ for } n \geq 2.$$

σ_n is decreasing function of $n (n \geq 2)$, we get

$$0 \le \sigma_n \le \sigma_2 = \frac{\prod_{j=1}^{s} \Gamma(\beta_j) \prod_{j=1}^{q} \Gamma(\alpha_j + A_j)}{\prod_{j=1}^{q} \Gamma(\alpha_j) \prod_{j=1}^{s} \Gamma(\beta_j + B_j)}.$$

$$\xi = 1 - \frac{2(1-\gamma)(1-\eta)}{(3-\gamma)(3-\eta)\sigma_2 - (1-\gamma)(1-\eta)},$$

where σ_2 is given by (5.2).This completes the proof of Theorem.

Theorem V.2. Let the functions

$$f_j(z) = z - \sum_{n=2}^{\infty} a_{n,j}\, z^n, \left(a_{n,j} \ge 0;\ (J = 1,2)\right)$$

be in the class $T^*(\gamma)$. Then $(f_1 * f_2)(z) \in T^*(\rho)$, where

$$\rho = 1 - \frac{2(1-\gamma)^2}{(3-\gamma)^2\sigma_2 - (1-\gamma)^2}$$

and σ_2 is given by (5.2).

Proof. if we set $\gamma = \eta$ in the above Theorem, the results follows.

Theorem V.3. Let the function $f(z)$ defined by (1.6) be in the class $T^*(\gamma)$ and also let

$$g(z) = z - \sum_{n=2}^{\infty} b_n z^n \quad, \text{for } |b_n| \le 1\ . \tag{5.12}$$

Then $(f * g)(z) \in T^*(\gamma)$.

Proof. By using (2.1), we get

$$\sum_{n=2}^{\infty} \sigma_n (2n-1-\gamma)|a_n b_n| = \sum_{n=2}^{\infty} \sigma_n (2n-1-\gamma) a_n |b_n|$$

$$\le \sum_{n=2}^{\infty} \sigma_n (2n-1-\gamma) a_n,$$

$$\le (1-\gamma),$$

where σ_n is given by (5.2). Hence it follows that $(f * g)(z) \in T^*(\gamma)$

Theorem 5.4. Let the functions

$$f_j(z) = z - \sum_{n=2}^{\infty} a_{n,j}\, z^n\ , (j = 1,2) \tag{5.13}$$

be in the class $T^*(\gamma)$. Then the function $h(z)$ defined by

$$h(z) = z - \sum_{n=2}^{\infty} (a_{n,1}^2 + a_{n,2}^2) z^n \tag{5.14}$$

is in the class $T^*(\Delta)$, where

$$\Delta = 1 - \frac{4(1-\gamma)^2}{(3-\gamma)^2\sigma_2 - 2(1-\gamma)^2} \tag{5.15}$$

and σ_2 is given by (5.2).

Proof. By using theorem 2.1, it is sufficient to prove that

$$\sum_{n=2}^{\infty} \sigma_n \frac{(2n-1-\Delta)}{(1-\Delta)} (a_{n,1}^2 + a_{n,2}^2) \le 1, \tag{5.16}$$

where $f_j \in T^*(\gamma_j)$ $(j = 1,2)$. Then

$$\sum_{n=2}^{\infty} \sigma_n \left(\frac{2n-1-\gamma}{1-\gamma}\right) a_{n,1} \le 1,$$

$$\sum_{n=2}^{\infty}\left[\sigma_n \frac{(2n-1-\gamma)}{(1-\gamma)}\right]^2 a_{n,1}^2 \le 1, \tag{5.17}$$

And $\sum_{n=2}^{\infty}\sigma_n\left(\frac{2n-1-\gamma}{1-\gamma}\right)a_{n,2} \le 1,$

$$\sum_{n=2}^{\infty}\left[\sigma_n \frac{(2n-1-\gamma)}{(1-\gamma)}\right]^2 a_{n,2}^2 \le 1, \tag{5.18}$$

then

$$\sum_{n=2}^{\infty}\frac{1}{2}\left[\sigma_n \frac{(2n-1-\gamma)}{(1-\gamma)}\right]^2 \left(a_{n,1}^2 + a_{n,2}^2\right) \le 1. \tag{5.19}$$

By comparing (5.16) and (5.19), we obtain

$$2\left(\frac{2n-1-\Delta}{1-\Delta}\right) = \sigma_n\left(\frac{2n-1-\gamma}{1-\gamma}\right)^2,$$

therefore,

$$\Delta(2n-1-\gamma)^2\sigma_n - 2(1-\gamma)^2 = (2n-1-\gamma)^2\sigma_n - 2(2n-1)(1-\gamma)^2,$$

$$\Delta = 1 - \frac{4(n-1)(1-\gamma)^2}{(2n-1-\gamma)^2\sigma_n - 2(1-\gamma)^2}.$$

$$\Delta = 1 - \frac{4(1-\gamma)^2}{(3-\gamma)^2\sigma_2 - 2(1-\gamma)^2}, n \ge 2,$$

which proves the Theorem.

VI. INTEGRAL TRANSFORM OF THE CLASS $T^*(\gamma)$

Let the function $f(z) \in T^*(\gamma)$. Then The integral transform

$$V_\lambda(f)(z) = \int_0^1 \lambda(t)\frac{f(tz)}{t}dt \ , \tag{6.1}$$

where λ is a real valued, non negative weight function normalized so that

$$\int_0^1 \lambda(t)\,dt = 1. \tag{6.2}$$

Since special cases of $\lambda(t)$ are particularly interesting such as

$$\lambda(t) = (1+c)t^c, \ \ c > -1, \tag{6.3}$$

for which V_λ is known as Bernadi operator and

$$\lambda(t) = \frac{(1+c)^\delta}{\Gamma(\delta)}t^c\left(\log\frac{1}{t}\right)^{\delta-1}, \qquad c > -1, \delta \ge 0 \tag{6.4}$$

this gives the Komatu operator (cf. [12]).

ThoremVI.1. Let the function $f \in T^*(\gamma)$, then $V_\lambda(f) \in T^*(\gamma)$.

Proof. By using (6.1) and (6.4), we have

$$V_\lambda(f) = \frac{(c+1)^\delta}{\Gamma(\delta)}\int_0^1 (-1)^{\delta-1}t^c(\log t)^{\delta-1}\left(z - \sum_{n=2}^{\infty}a_n z^n t^{n-1}\right)dt \ ,$$

$$= \frac{(c+1)^\delta(-1)^{\delta-1}}{\Gamma(\delta)}\left[\int_0^1 t^c(\log t)^{\delta-1}\left(z - \sum_{n=2}^{\infty}a_n z^n t^{n-1}\right)\right],$$

$$V_\lambda(f) = z - \sum_{n=2}^{\infty}\left(\frac{c+1}{c+n}\right)^\delta a_n z^n \ , \tag{6.5}$$

by (2.1), $f \in T^*(\gamma)$, if and only if

$$\sum_{n=2}^{\infty} \frac{(2n-1-\gamma)}{(1-\gamma)} \left(\frac{\prod_{j=1}^{s} \Gamma(\beta_j) \prod_{j=1}^{q} \Gamma(\alpha_j + A_j(n-1))}{\prod_{j=1}^{q} \Gamma(\alpha_j) \prod_{j=1}^{s} \Gamma(\beta_j + B_j(n-1))\,(n-1)!} \right) \left(\frac{c+1}{c+n}\right)^{\delta} a_n \le 1,$$

since $\frac{c+1}{c+n} < 1$, therefore,

$$\sum_{n=2}^{\infty} \frac{(2n-1-\gamma)}{(1-\gamma)} \left(\frac{\prod_{j=1}^{s} \Gamma(\beta_j) \prod_{j=1}^{q} \Gamma(\alpha_j + A_j(n-1))}{\prod_{j=1}^{q} \Gamma(\alpha_j) \prod_{j=1}^{s} \Gamma(\beta_j + B_j(n-1))\,(n-1)!} \right) a_n \le 1.$$

Proof is completes.

Theorem 6.2. Let the function $f \in T^*(\gamma)$. Then $V_\lambda(f)$ is starlike of order $\zeta\ (0 \le \zeta < 1)$ in $|z| < R_1$, where

$$R_1 = \inf_n \left[\left(\frac{c+n}{c+1}\right)^{\delta} \frac{(1-\zeta)(2n-1-\gamma)\sigma_n}{(n-\zeta)(1-\gamma)} \right]^{1/n-1} \qquad (6.6)$$

and σ_n is given by (1.11).

Proof. It suffices to prove that

$$\left| \frac{z(V_\lambda(f)(z))'}{V_\lambda(f)(z)} - 1 \right| < 1-\zeta, \qquad (6.7)$$

$$\left| \frac{-\sum_{n=2}^{\infty} (n-1)\left(\frac{c+1}{c+n}\right)^{\delta} a_n z^{n-1}}{1-\sum_{n=2}^{\infty} \left(\frac{c+1}{c+n}\right)^{\delta} a_n z^{n-1}} \right| < 1-\zeta,$$

therefore

$$\sum_{n=2}^{\infty} (n-1)\left(\frac{c+1}{c+n}\right)^{\delta} a_n |z|^{n-1} \le 1-\zeta-(1-\zeta)\sum_{n=2}^{\infty} \left(\frac{c+1}{c+n}\right)^{\delta} a_n |z|^{n-1},$$

$$\sum_{n=2}^{\infty} a_n |z|^{n-1} \le \left(\frac{c+n}{c+1}\right)^{\delta} \frac{(1-\zeta)}{(n-\zeta)}, \qquad (6.8)$$

(6.8) is true if

$$|z| \le \left\{ \left(\frac{c+n}{c+1}\right)^{\delta} \frac{(1-\zeta)(2n-1-\gamma)\sigma_n}{(n-\zeta)(1-\gamma)} \right\}^{\frac{1}{n-1}}, n \ge 2, \qquad (6.9)$$

where σ_n is defined by (1.11).

Proof of the Theorem is completed.

Theorem 6.3. Let the function $f \in T^*(\gamma)$. Then $V_\lambda(f)$ is convex of order $\zeta\ (0 \le \zeta < 1)$ in $|z| < R_2$, where

$$R_2 = \inf_n \left[\left(\frac{c+n}{c+1}\right)^{\delta} \frac{(1-\zeta)(2n-1-\gamma)}{n(n-\zeta)(1-\gamma)} \sigma_n \right]^{1/n-1}, \qquad (6.10)$$

and σ_n is defined by (1.11).

Proof. The proof of the Theorem can be obtained as Theorem 6.2.

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